THE LAST LECTURE

☐ Exam 1 scheduled
☐ Exam 2 l8r
☐ Homework solutions are up!
☐ Record/mic

1. SLICE RIBBON CONJECTURE

Recall: A knot K is ribbon if it has a smooth slice disk with no local maxima (equiv. bounds a disk in S^3 with ribbon ∩'s)

KNOWN: Ribbon ⟹ Slice

(Fox '60s) Slice ⟹ Ribbon

Some thoughts:

- true for some families (2-bridge knots, odd 3-pretzels, torus, …)
  Lisca Greene-Jabuka good exercise!
- ∃ non-ribbon knots w/ unknown slice status
- not all slice disks are *themselves* ribbon disks

2. SMOOTH SLICENESS (of certain knots)

2A. ∃ Knots with unknown smooth slice status:

a. Wh^+() or more generally Wh^+ of any negative torus knot

b. $Wh^\pm$()

2B. K slice iff Wh(K) slice

Known for any families?
Do we know of Wh(non-slice) that is slice?

What category…..? False in TOP

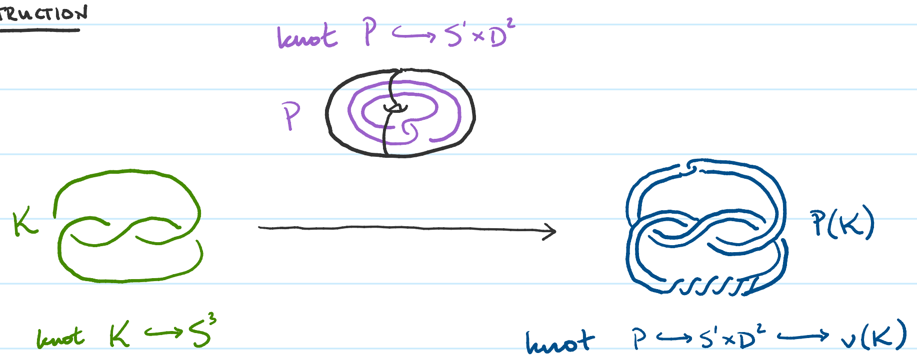
known: ⟹ (Why? Double slice disk and band)

open: ⟸ (THOUGHTS: already hard to see if Wh(·) is slice)

2C. More generally: $K \sim J \Rightarrow Wh(K) \sim Wh(J)$

so $\exists$ function $Wh: \mathcal{C} \longrightarrow \mathcal{C}$ (not a homomorphism! Injective? Surjective?)

SATELLITE CONSTRUCTION



so P defines a function $\mathcal{C} \longrightarrow \mathcal{C}$ $[K] \longmapsto [P(K)]$

(When is P injective/surjective)

OPEN Conjecture (Hedden) The only homomorphisms on $\mathcal{C}$ are:

- Trivial $[K] \longmapsto 0$
- Identity $[K] \longmapsto [K]$
- Reverse $[K] \longmapsto [-K]$

"What does this say about the group str of $\mathcal{C}$?

3. GROUP STRUCTURE OF $\mathcal{C}_{SM/TOP}$

"Throw your favorite question at $\mathcal{C}$ and it's probably open"

KNOWN $\exists\ \mathbb{Z}^{\infty} \underset{\curvearrowleft}{\longrightarrow} \mathcal{C}$ coming from ε-invariant (Hom)

$\exists\ \mathbb{Z}_2^{\infty} \longrightarrow \mathcal{C}$ coming from 4_1 = 

(Why?)

$\underbrace{4_1 = m(4_1)}_{\text{Defn amphichiral}}$ so $4_1 \# 4_1 \sim$ unknot

OPEN $\mathbb{Q} \longrightarrow \mathcal{C}$?

$\mathbb{Z}_n \longrightarrow \mathcal{C}$ for all n?

Is all 2-torsion from amphichiral knots?

Does (a generalization of) the s-inv't detect a $\mathbb{Z}^{\infty}$ summand?

Open problems, part 2

First, a problem related to a big open problem in topology

Question: Are ribbon disc complements aspherical?

[Recall a space X is aspherical if $\pi_i X = 0 \ \forall i \geq 2$]

a special case of: Whitehead (asphericity) conjecture 1941:

Every connected subcx of a 2-dim aspherical CW cx is itself aspherical.

Note: Ribbon disc complements are 2D cw cxs

[add in a 2-cell to get B^4, which is aspherical]

: Slice disc complements need not be aspherical.

[e.g. can always tube into a 2-knot. Then π_2 will include π_2 of the 2-knot, which can be large]

: Known in some special cases, e.g. only two minima.

: Asphericity of knot complements was one of the original motivation

⟶

Next some questions related to general 3-/4-mfds

Recall, we have constructions:

$$\begin{tikzcd} \dfrac{K \subseteq S^3 \text{ knot}}{\text{concordance}} \arrow[r, bend left, "p/q \text{ Dehn surgery}"] \arrow[dr, bend left, "p\text{-fold cyclic br. cover}", "p \text{ prime power}"'] & S^3_{p/q}(K), \quad H_*(S^3_{p/q}(K);\mathbb{Z}) \cong H_*(L(p,1);\mathbb{Z}) \Big/ \mathbb{Z}H_* \text{ cobordi} \\ & \Sigma_p(K), \quad H_*(\Sigma_p(K);\mathbb{Q}) \cong H_*(S^3;\mathbb{Q}) \Big/ \mathbb{Q}H_* \text{ cobordin} \end{tikzcd}$$

Vague question: To what extent do these 3-mflds recover the concordance class?

Specific question: Akbulut-Kirby conjecture 1978:

If ~~∃~~ $S^3_0(K) \underset{\text{homeo/diffeo}}{\approx} S^3_0(J)$ is K $\underset{\substack{\uparrow \\ \text{sm or TOP}}}{\text{conc.}}$ to J ?

Answer: No in sm. category [Yasui, 2017]

: Open in TOP.

Note: if K TOP slice, then true in TOP

if K sm slice, then true in sm category if 4DPC is true.

Analogous questions in 3D knot theory: $S^3_0(K) \approx S^1 \times S^2 \Leftrightarrow K = \mathcal{U}$ [Gabai]

$\Sigma_p(K) \approx S^3 \Leftrightarrow K = \mathcal{U}$

Smith conjecture

More generally, can consider $\mathcal{C} \longrightarrow \prod_{p/q} S^3_{p/q}(K)$ or $\mathcal{C} \rightarrow \prod_p \Sigma_p(K)$ and ask whether injective, size of kernel, etc.

$\longrightarrow$

Next two are going to ask about more general notions of slice and concordance

Given $K: S^1 \hookrightarrow Y = \partial W^4$ we say K is slice in W if

$$\begin{tikzcd} S^1 \arrow[r, hook, "K"] \arrow[d, hook, "\partial"'] & Y \arrow[d, hook, "\partial"] \\ D^2 \arrow[r, hook, "\Delta"] & W \end{tikzcd}$$

sm/top

Question: Does there exist $K \subseteq S^3$ and W^4 s.t.

$H_*(W;\mathbb{Z}) = H_*(B^4;\mathbb{Z})$ i.e. W is an $\mathbb{Z}H_*B$,

s.t. K is slice in W but not slice in B^4?

This is interesting because:

- none of our known invts seem to be able to detect the difference i.e. all the invts mentioned in class have the same behaviour for "homology slice" knots as "slice knots

$\boxed{\text{EXCEPT}}$ the s-invt

Pipe dream: 1. Construct a smooth, compact W^4 s.t. $W \simeq B^4$, $\partial W = S^3$ and a knot $K \subseteq S^3$ s.t. K sm. slice in W.

2. Compute $s(K) \neq 0$

[i.e. disprove 4DPC]

[Alternatively try similar strategies to find exotic sm str. on other 4-mflds]

In a similar vein, can ask a more relative version of the question:

Let $K \subseteq Y^3$ where $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$, i.e. Y is an $\mathbb{Z}H_*S^3$.

Suppose that Y is $\mathbb{Z}H_*$ cobordant to S^3, i.e. $\exists W^4$ with $\partial W = Y \sqcup -S^3$

and $i_*: H_i(Y;\mathbb{Z}) \xrightarrow{\cong} H_i(W;\mathbb{Z})$

& $i_*: H_i(S^3;\mathbb{Z}) \xrightarrow{\cong} H_i(W;\mathbb{Z})$

<u>Question</u>: Is K concordant in W to some $J \subseteq S^3$?

Or, does there exist a $\mathbb{Z}H_*$ cob V between Y & S^3 s.t. K is concordant in V to some $J \subseteq S^3$?

Answer: No in smooth category [Akbulut, Levine]

Open in TOP category.

Note: none of our known TOP invariants seem able to tell the differe

$\longrightarrow$

Finally a question which seems perhaps a bit specialized at first

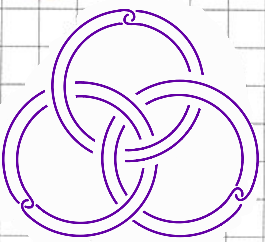
Recall: we saw that knots with Alexander polynomial one are TOP slice.

Vague question: What is the analogue for links?

[Recall, a link is an embedding $\bigsqcup_{i=1}^{n} S^1 \hookrightarrow S^3$]

Answer: "Good boundary links",

e.g. Wh(Borromean rings)



[Boundary link: comps bound disjoint Seifert surfaces in S^3]

["good" refers to a certain π_1-condition]

Question: Are good boundary links slice? i.e. freely slice? i.e. $\pi_1(B^4 \setminus \sqcup \Delta^2) \cong$ free?

$$\begin{tikzcd} L: \sqcup S^1 \arrow[r, hook] \arrow[d, hook, "\partial"'] & S^3 \arrow[d, hook, "\partial"] \\ \sqcup D^2 \arrow[r, hook, "\Delta", "TOP"'] & B^4 \end{tikzcd}$$

Note: We used surgery theory to show Alex poly one knots are TOP slice.

- <u>But</u> there was a π_1- restriction in the big theorem of Freedman-Quinn we used.
- for GBLs we would need a version for free gps, which is not known
  - if we did, could do TOP surgery for all gps.

$\boxed{\text{Surprise fact}}$ the converse is also true!

i.e. if we show that GBLs are freely slice, then that implies that TOP surgery holds for all